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DYNAMICS OF STRUCTURAL SYSTEMS WITH SPATIAL RANDOMNESS

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Abstract—The objective of this paper is the analysis of the dynamics of structural systems with randomly varying parameters. The formulation and analysis are based on the theory of random integral equations. The general analysis (including error estimation) is applied to the problems of harmonic vibration of (i) a beam with randomly varying density and (ii) a uniform beam resting on a randomly inhomogeneous Winkler foundation. For these applications, the analytical formulae for the mean and correlation function of the response are illustrated numerically and graphically.

1. INTRODUCTION

In traditional analysis of structural systems (e.g. beams, plates, shells, etc.), it is usually assumed that the material and geometrical properties of the system in question are constant. Although such an assumption has long been a basis for numerous practical solutions, it is evident that some amount of randomness in characterization of the system properties (e.g. mass density, cross section, bending rigidity) is unavoidable. In general, models for the behavior of structural elements should take into account uncertainty and inhomogeneity in their mechanical properties. Such models become especially important now when modern engineering structures are becoming increasingly complex and contain more technologically advanced materials (e.g. composites).

As a result, an increasing amount of attention has been devoted to various structural analysis problems in which spatial randomness occurs in the system properties. Apparently, the first static problems of this kind were connected with geotechnical applications, e.g. beams resting on statistically inhomogeneous Winkler foundation (Bolotin, 1965, 1971), and were treated by perturbation techniques. Subsequently, Monte Carlo simulation (Astil *et al.*, 1972), Green's function approaches (Bucher and Shinozuka, 1988; Kardara *et al.*, 1989) and the stochastic finite element method (Vanmarke and Grigoriu, 1983; Deodatis and Shinozuka 1988; Ghanem and Spanos 1991; and the references therein) were proposed.

Vibration analysis of statistically inhomogeneous structural systems was initiated many years ago—mostly through the use of perturbation techniques [cf., Boyce (1962, 1967); Collins and Thomson, (1969), Sobczyk (1970, 1972)]. The dynamic problems for the systems considered were mainly focused on characterization of random eigenvalues [in this context, see for example the book of vom Scheidt and Purkert (1983) and the recent paper by Iyengar and Manohar (1989)]. Dynamic behavior of a long (infinite) beam resting on a randomly varying foundation has recently been treated by the perturbation and finite element techniques [cf. Fryba *et al.* (1993)]. It is worth adding that spatial randomness has been previously treated in other fields; for example in the analysis of wave propagation [cf., Sobczyk (1985)].

Although the literature devoted to problems of spatial variability in structural systems is quite extensive, it mainly deals with perturbation and finite element techniques. However, it turns out that a certain class of dynamic problems with spatially random parameters (especially, the dynamic analysis of beams with random properties) can effectively and rigorously be analyzed via a random integral equation formulation. Such an approach constitutes a clear analytical approximate method with defined range of applicability and with specific error estimates that depend explicitly on the intensity of the spatial randomness. The objective of this paper is to provide an analysis along these lines.

This paper deals with the analysis of the dynamics of structural elements with random properties based on the theory of random integral equations [cf., Tsokos and Padgett (1974); Szynal and Wedrychowicz (1985, 1988)]. First, the necessary mathematical theory, including the successive approximations of the solution and the error estimation, is briefly presented. Then, the analysis of the harmonic vibration of beams with spatial randomness is performed, and graphical and numerical results are presented.

2. GENERAL FORMULATION

The dynamics of a wide class of elastic structural systems with spatially distributed randomness are governed by the equations of form

$$L_{\mathbf{r}}w + A(\mathbf{r},\gamma)\frac{\partial^2 w}{\partial t^2} = q(\mathbf{r},t) \tag{1}$$

where $\mathbf{r}$ denotes the spatial variable, in general, $\mathbf{r} = (x, y, z)$; $(\Gamma, \mathscr{F}, P)$ is the basic probability space, and $\gamma \in \Gamma$ represents an elementary event; for each $\gamma \in \Gamma$ eqn (1) is a possible realization of the process in question. The random properties of the system are jointly characterized by the random field $A(\mathbf{r}, \gamma)$; $q(\mathbf{r}, t)$ is an external excitation (which can be considered deterministic or random), $w = w(\mathbf{r}, t)$ is the unknown displacement field, $L_{\mathbf{r}}$ is a deterministic differential operator with respect to spatial variables. Equation (1) has to be implemented by the appropriate initial and boundary conditions, which we shall represent symbolically as

$$w(\mathbf{r}, t_0) = w_0(\mathbf{r}), \quad \frac{\partial}{\partial t}w(\mathbf{r}, t_0) = w_1(\mathbf{r}), \quad Hw(\mathbf{r}, t) = 0, \quad \mathbf{r} \in S \tag{2}$$

where S denotes the boundary of the structural element, and H is an appropriate operator acting on $w(\mathbf{r}, t)$ when $\mathbf{r} \in S$.

In the case when the system considered is an elastic plate or elastic beam, the random field $A(\mathbf{r}, \gamma)$ has the following form:

$$A(\mathbf{r}, \gamma) = \{\rho(\mathbf{r})h(\mathbf{r})D^{-1}\}(\gamma) \tag{3}$$

where $\rho(\mathbf{r})$ denotes the density of the material, $h(\mathbf{r})$ is the cross sectional area, and D is the bending rigidity (the product of the mass moment of inertia I and the Young's modulus E). For the method presented herein, D should be a deterministic constant or a random variable. For real structural elements

$$\rho > 0, \quad h > 0, \quad D > 0. \tag{4}$$

Let us consider the harmonic vibrations when the excitation and the response are periodic in time. If, for instance, $q(\mathbf{r}, t) = Q(\mathbf{r}) \sin pt$ and $w(\mathbf{r}, t) = Y(\mathbf{r}) \sin pt$, then equation (1) takes the form

$$L_{\mathbf{r}} Y - p^2 A(\mathbf{r}, \gamma) Y = Q(\mathbf{r}) \tag{5}$$

with the boundary conditions (2). Here, D^{-1} is also included as a multiplicative constant in $Q(\mathbf{r})$. The above equation can be rewritten as

$$L_{\mathbf{r}} Y = p^2 A(\mathbf{r}, \gamma) Y + Q(\mathbf{r}) \tag{6}$$

and represented (along with the boundary conditions) in the form of the random integral equation

$$Y(\mathbf{r}) = \varphi(\mathbf{r}) + \int_B K(\mathbf{r}, \mathbf{r}', \gamma) Y(\mathbf{r}') \, \mathrm{d}\mathbf{r}' \tag{7}$$

where B is a domain occupied by the structural element,

$$\varphi(\mathbf{r}) = \int_B Q(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \, \mathrm{d}\mathbf{r}' \tag{8}$$

$$K(\mathbf{r}, \mathbf{r}', \gamma) = p^2 A(\mathbf{r}', \gamma) G(\mathbf{r}, \mathbf{r}') \tag{9}$$

and $G(\mathbf{r}, \mathbf{r}')$ is the Green's function associated with operator $L_{\mathbf{r}}$ and specified boundary conditions. The Green's function $G(\mathbf{r}, \mathbf{r}')$ determines the static displacement of the element at point $\mathbf{r}$ due to the unit force acting at point $\mathbf{r}'$. For many structural elements $G(\mathbf{r}, \mathbf{r}')$ can be determined analytically.

Equations (7)–(9) constitute the random integral equation formulation of the problem. We wish to determine the unknown random field $Y(\mathbf{r})$; as formulae (8) and (9) indicate, the inhomogeneous term $\varphi(\mathbf{r})$ and random kernel $K(\mathbf{r}, \mathbf{r}', \gamma)$ are given.

2.1. *Remarks*

In some situations, especially when the spatial randomness is characterized by a random field with constant mean value m_A, one may modify the analysis in the following way. Let

$$A(\mathbf{r}, \gamma) = m_A + A'(\mathbf{r}, \gamma) \tag{10}$$

where $A'(\mathbf{r}, \gamma)$ characterizes the random fluctuations of the process about m_A. Introducing the new operator

$$L'_{\mathbf{r}} = L_{\mathbf{r}} - p^2 m_A \tag{11}$$

equation (5) then becomes

$$L'_{\mathbf{r}} Y - p^2 A'(\mathbf{r}, \gamma) Y = Q(\mathbf{r}) \tag{12}$$

where the mean of $A'(\mathbf{r}, \gamma)$ is zero. Therefore, the associated random integral equation takes the form of (7), whereas in formulae (8) and (9), $G(\mathbf{r}, \mathbf{r}')$ is replaced by the Green's function corresponding to the operator $L'_{\mathbf{r}}$ given by (11); and $A(\mathbf{r}', \gamma)$ in (9) is replaced by $A'(\mathbf{r}', \gamma)$. Of course, in all subsequent formulae, m_A should be replaced by $m_{A'} = 0$. It is seen that in the modification indicated here, the effect of the mean value of the random inhomogeneity is incorporated into the Green's function (which is associated with operator (11)). In this way, the Green's function is a little more complex, but the formulae for the moment response includes fewer terms. In addition, the applicability range of the method is significantly expanded.

It is clear that the formulation of the problem presented above, both in its standard and modified form, includes as a special case, harmonic vibration of an elastic plate and beam resting on statistically inhomogeneous Winkler foundation. In this case, the random function $A(\mathbf{r},\gamma)$ is the difference $A(\mathbf{r},\gamma) = A_b(\mathbf{r},\gamma) - A_f(\mathbf{r},\gamma)$ where $A_b(\mathbf{r},\gamma)$ characterizes the properties of the structural element in question and $A_f(\mathbf{r},\gamma)$ describes the properties of the foundation.

3. ANALYSIS OF RANDOM INTEGRAL EQUATION

To formulate the basic statements on the existence and uniqueness of the solution of eqn (7) and its successive approximations we shall introduce first the necessary definitions and notations.

Let $L_2(\Gamma,\mathscr{F},P)$ denote, as usual, the space of square-integrable (with respect to P) random variables $X(\gamma)$ with the norm

$$\|X(\gamma)\|_{L_2} = \langle |X(\gamma)|^2\rangle^{1/2} = \left\{\int_\Gamma |X(\gamma)|^2\,\mathrm{d}P(\gamma)\right\}^{1/2}. \tag{13}$$

Let the kernel $K(\mathbf{r},\mathbf{r}',\gamma)$ be a measurable (e.g. continuous) function which maps the domain $B\times B$ into $L_2(\Gamma,\mathscr{F},P)$. Random function $Y(\mathbf{r},\gamma)$ is a solution of eqn (7) if, for each fixed $\mathbf{r}\in B$, $Y(\mathbf{r},\gamma)\in L_2(\Gamma,\mathscr{F},P)$ and satisfies eqn (7) almost surely (with respect to P).

Let $C(B, L_2(\Gamma,\mathscr{F},P))$, that is, C denotes the space of all continuous and bounded functions on B with values in $L_2(\Gamma,\mathscr{F},P)$ with the topology defined by the norm:

$$\|Y\| \equiv \sup_{\mathbf{r}\in B}\|Y(\mathbf{r},\gamma)\|_{L_2}. \tag{14}$$

Such a definition of the norm in C implies that if $Y(\mathbf{r},\gamma)\in C$ then for each $\mathbf{r}\in B$

$$\|K(\mathbf{r},\mathbf{r}',\gamma)\,Y(\mathbf{r},\gamma)\|_{L_2} \leqslant \|K(\mathbf{r},\mathbf{r}',\gamma)\|_{L_2}\,\|Y(\mathbf{r},\gamma)\|_{L_2}. \tag{15}$$

The basic statement (theorem) concerning eqn (7) is as follows: if
(i)

$$\eta = \sup_{\mathbf{r}\in B}\left[\int_B \|K(\mathbf{r},\mathbf{r}',\gamma)\|_{L_2}\,\mathrm{d}\mathbf{r}'\right] < 1 \tag{16}$$

(ii)

$$\sup_{\mathbf{r}\in B}\|\varphi(\mathbf{r},\gamma)\|_{L_2} < \infty \tag{17}$$

then there exists one and only one solution $Y(\mathbf{r},\gamma)\in C$ of eqn (7).

The proof makes use of relations (15)–(17) and the Banach fixed point principle [cf., Tsokos and Padgett, (1974); Szynal and Wedrychowicz, (1988)]. The sequence of the successive approximations:

$$\begin{aligned} Y_0(\mathbf{r},\gamma) &= \varphi(\mathbf{r},\gamma),\\ Y_{n+1}(\mathbf{r},\gamma) &= \varphi(\mathbf{r},\gamma) + \int_B K(\mathbf{r},\mathbf{r}',\gamma)\,Y_n(\mathbf{r}',\gamma)\,\mathrm{d}\mathbf{r}, \quad n = 0,1,2,\ldots \end{aligned} \tag{18}$$

converges to the solution of eqn (7) in space $C(B, L_2(\Gamma,\mathscr{F},P))$. Indeed, for $\mathbf{r}\in B$

$$\begin{aligned}\| Y_{n+1}(\mathbf{r},\gamma) - Y_n(\mathbf{r},\gamma)\| &= \sup_{\mathbf{r}} \| Y_{n+1}(\mathbf{r},\gamma) - Y_n(\mathbf{r},\gamma)\|_{L_2} \\ &= \sup_{\mathbf{r}} \left\| \int_B K(\mathbf{r},\mathbf{r}',\gamma) Y_n(\mathbf{r}')\,\mathrm{d}\mathbf{r}' - \int_B K(\mathbf{r},\mathbf{r}',\gamma) Y_{n-1}(\mathbf{r}',\gamma)\,\mathrm{d}\mathbf{r}' \right\|_{L_2} \\ &\leqslant \sup_{\mathbf{r}} \int_B \|K(\mathbf{r},\mathbf{r}',\gamma)[Y_n(\mathbf{r}') - Y_{n-1}(\mathbf{r}')]\|_{L_2}\,\mathrm{d}\mathbf{r}' \\ &\leqslant \sup_{\mathbf{r}} \left\{ \| Y_n(\mathbf{r}) - Y_{n-1}(\mathbf{r})\| \int_B \|K(\mathbf{r},\mathbf{r}',\gamma)\|_{L_2}\,\mathrm{d}\mathbf{r}' \right\} \\ &\leqslant \eta \| Y_n(\mathbf{r}) - Y_{n-1}(\mathbf{r})\|. \end{aligned} \tag{19}$$

Repeating the above estimation $n-1$ times, we obtain

$$\| Y_{n+1}(\mathbf{r},\gamma) - Y_n(\mathbf{r},\gamma)\| \leqslant \eta^n \| Y_1(\mathbf{r},\gamma) - Y_0(\mathbf{r},\gamma)\|. \tag{20}$$

It is clear that (with use of the triangle inequality)

$$\begin{aligned}\| Y_{n+k} - Y_n\| &\leqslant \| Y_{n+k} - Y_{n+k-1}\| + \| Y_{n+k-1} - Y_n\| \\ &\leqslant \| Y_{n+k} - Y_{n+k-1}\| + \| Y_{n+k-1} - Y_{n+k-2}\| + \cdots + \| Y_{n+2} - Y_{n+1}\| + \| Y_{n+1} - Y_n\|\end{aligned}$$

by virtue of (20)

$$\begin{aligned}\| Y_{n+k} - Y_n\| &\leqslant \eta^n \| Y_1 - Y_0\| + \eta^{n+1}\| Y_1 - Y_0\| + \cdots + \eta^{n+k-2}\| Y_1 - Y_0\| \\ &\quad + \eta^{n+k-1}\| Y_1 - Y_0\| = \| Y_1 - Y_0\|(\eta^n + \eta^{n+1} + \cdots + \eta^{n+k-1}).\end{aligned}$$

Since η is assumed to be less than one, we have

$$\| Y_{n+k} - Y_n\| \leqslant \eta^n \frac{1-\eta^k}{1-\eta} \| Y_1 - Y_0\| \leqslant \frac{\eta^n}{1-\eta} \| Y_1 - Y_0\|.$$

Since k is an arbitrary natural number, the sequence $\{Y_n(\mathbf{r},\gamma)\}$ is a Cauchy sequence in space C, so it converges to the element $Y(\mathbf{r},\gamma) \in C$, and

$$\| Y(\mathbf{r},\gamma) - Y_n(\mathbf{r},\gamma)\| \leqslant \frac{\eta^n}{1-\eta} \| Y_1 - Y_0\| \leqslant \frac{\eta^{n+1}}{1-\eta} \|\varphi\|. \tag{21}$$

Inequality (21) gives the error of approximation of solution $Y(\mathbf{r},\gamma)$ by $Y_n(\mathbf{r},\gamma)$ for each $\mathbf{r} \in B$. Since for each $\mathbf{r} \in B$ the values of $Y(\mathbf{r},\gamma)$ belong to $L_2(\Gamma, \mathscr{F}, P)$, this means that we have, for each $\mathbf{r} \in B$, a bound

$$\langle | Y(\mathbf{r},\gamma) - Y_n(\mathbf{r},\gamma)|^2 \rangle^{1/2} \leqslant \frac{\eta^{n+1}}{1-\eta} \|\varphi\|. \tag{22}$$

3.1. *Remark*

In addition to the continuous approximation of the solution $Y(\mathbf{r},\gamma)$ via successive approximations it may be useful (in dealing with applications) to consider the discrete representation (approximation) of random integral equation (6). For instance, if the problem is spatially one-dimensional ($\mathbf{r} = x \in [[0, l]]$), we consider the discrete points: $0 = x_0 < x_1 < \ldots$, and $x_i - x_{i-1} = r$, $i = 1, 2, \ldots$, and $x_n = x_0 + nr = nr$. For fixed $x = x_n$, the interval $[0, l]$ is divided into n subintervals. As $r \to 0$, then for fixed x such that

$x = x_n = nr$ we must have $n \to \infty$. The discrete version of integral equation (6) is [cf., Tsokos and Padgett, (1974); Sobczyk, (1991)]

$$Y_n = \varphi(x_n) + \sum_{i=0}^{n} W_{n,i} K_{n,i}(\gamma) Y_i \equiv U Y_i \tag{23}$$

where $K_{n,i}(\gamma) = K(x_n, \xi_i, \gamma)$, $Y_i = Y(x_i, \gamma)$, and $W_{n,i}$ are appropriate weights (e.g. such as in the composite trapezoidal rule). In this case, the sequence of successive (algebraic) approximations $\tilde{Y}_n(\gamma)$ of $Y_n(\gamma)$ for each $x = x_n$ takes the form

$$\begin{aligned} \tilde{Y}_n^0(\gamma) &= \varphi(x_n, \gamma), \\ \tilde{Y}_n^{(k+1)}(\gamma) &= U Y_n^{(k)}(\gamma), \quad k = 0, 1, \ldots \end{aligned} \tag{24}$$

It has been proven that the above sequence of approximations converges to the true solution $Y(x, \gamma)$ at $x = x_n$ as $k \to \infty$.

4. APPLICATIONS

4.1. *Elastic beam with random mass density*

Let us consider vibrations of a finite elastic beam with randomly varying density; the bending rigidity is assumed here to be constant. In this case the basic differential equation (1) reduces to the following one:

$$\frac{\partial^4 w(x,t)}{\partial x^4} + A(x,\gamma)\frac{\partial^2 w}{\partial t^2} = q(x,t) \tag{25}$$

where $q(x, t)$ includes factor D^{-1}, and according to (3),

$$A(x,\gamma) = hD^{-1}\rho(x,\gamma). \tag{26}$$

It should be noted that D can be treated as a random variable as well. Here, randomness of D is accounted for by considering the right hand side of (25), and consequently $\varphi(x)$, to be random.

Of the various possible boundary conditions at the ends of the beam we take here those corresponding to simply supported ends, that is

$$w(x,t)|_{x=0,l} = 0, \quad \frac{\partial^2 w}{\partial x^2}(x,t)|_{x=0,l} = 0. \tag{27}$$

Therefore, the periodic (in time) vibrations are governed by the random integral equation (7)

$$Y(x) = \varphi(x) + \int_0^l K(x,\xi,\gamma) Y(\xi)\, \mathrm{d}\xi \tag{28}$$

where $\varphi(x)$ and $K(x, \xi, \gamma)$ are given by formulae (8) and (9) where $\mathbf{r} = x$, $\mathbf{r}' = \xi$, and $B = [0, l]$. The Green's function $G(x, \xi)$ accounting for boundary conditions (27) is known, namely

$$G(x,\xi) = \begin{cases} \dfrac{l^3}{6}\left(1-\dfrac{\xi}{l}\right)\dfrac{x}{l}\left[1-\left(\dfrac{x}{l}\right)^2-\left(1-\dfrac{\xi}{l}\right)^2\right], & x \leqslant \xi \\ \dfrac{l^3}{6}\left(1-\dfrac{x}{l}\right)\dfrac{\xi}{l}\left[1-\left(\dfrac{\xi}{l}\right)^2-\left(1-\dfrac{x}{l}\right)^2\right], & x \leqslant \xi. \end{cases} \tag{29}$$

The first terms of the sequence of successive approximations take the form

$$\begin{aligned} Y_0(x,\gamma) &= \varphi(x,\gamma) \\ &= \int_0^l Q(\xi,\gamma)G(x,\xi)\,\mathrm{d}\xi, \\ Y_1(x,\gamma) &= \varphi(x,\gamma)+\int_0^l K(x,\xi,\gamma)\varphi(\xi,\gamma)\,\mathrm{d}\xi, \\ Y_2(x,\gamma) &= \varphi(x,\gamma)+\int_0^l K(x,\xi_1,\gamma)\varphi(\xi_1,\gamma)\,\mathrm{d}\xi_1 \\ &\quad +\int_0^l K(x,\xi_2,\gamma)\left[\int_0^l K(\xi_2,\xi_1,\gamma)\varphi(\xi_1,\gamma)\,\mathrm{d}\xi_1\right]\mathrm{d}\xi_2. \end{aligned} \tag{30}$$

$\vdots$

Let us assume that random function $A(x,\gamma)$ is determined by its mean and covariance function:

$$\langle A(x,\gamma)\rangle = m_A(x), \tag{31}$$

$$\langle [A(x_1,\gamma)-m_A(x_1)][A(x_2,\gamma)-m_A(x_2)]\rangle = R_A(x_1,x_2) = \sigma_A^2 r_A(x_1,x_2). \tag{32}$$

In addition, we shall denote

$$\hat{R}_A(x_1,x_2) = \langle A(x_1,\gamma)A(x_2,\gamma)\rangle.$$

It is clear that:

$$R_A(x_1,x_2) = \hat{R}_A(x_1,x_2)-m_A(x_1)m_A(x_2).$$

The mean values of the first approximations are as follows (we assume that $\varphi(x,\gamma)$ and $A(x,\gamma)$ are independent).

$$\begin{aligned} m_0(x) &= \langle Y_0(x,\gamma)\rangle = \langle\varphi(x,\gamma)\rangle \\ &= \int_0^l \langle Q(\xi,\gamma)\rangle G(x,\xi)\,\mathrm{d}\xi = m_\varphi(\xi) \end{aligned} \tag{33}$$

$$\begin{aligned} m_1(x) &= \langle Y_1(x,\gamma)\rangle \\ &= \langle\varphi(x,\gamma)\rangle+\int_0^l \langle K(x,\xi,\gamma)\rangle\langle\varphi(\xi,\gamma)\rangle\,\mathrm{d}\xi \\ &= m_\varphi(x)+p^2\int_0^l m_A(\xi)m_\varphi(\xi)G(x,\xi)\,\mathrm{d}\xi \end{aligned} \tag{34}$$

$$\begin{aligned} m_2(x) &= \langle Y_2(x,\gamma)\rangle \\ &= m_\varphi(x)+p^2\int_0^l m_A(\xi_1)m_\varphi(\xi_1)G(x,\xi_1)\,\mathrm{d}\xi_1 \\ &\quad +p^4\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2)m_\varphi(\xi_1)G(x,\xi_2)G(\xi_2,\xi_1)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \end{aligned} \tag{35}$$

In general, for $k = 1, 2, \ldots$

$$m_k(x) = \langle Y_k(x,\gamma)\rangle = m_\varphi(x)+b_1c_1(x)+b_2c_2(x)+\cdots+b_kc_k(x) \tag{36}$$

where

$$b_k = p^{2k}, \tag{37}$$

$$\begin{aligned} c_k(x) = \int_k \cdots \int \hat{R}_A^{(k)}(\xi_1,\xi_2,\ldots,\xi_k)m_\varphi(\xi_k)G(x,\xi_1)G(\xi_1,\xi_2)\ldots \\ \times G(\xi_{k-1},\xi_k)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\ldots\mathrm{d}\xi_k \end{aligned} \tag{38}$$

and for $k = 1, 2, \ldots$

$$\begin{aligned} \hat{R}_A^{(k)}(\xi_1,\xi_2\ldots\xi_k) &= \langle A(\xi_1,\gamma)A(\xi_2,\gamma)\ldots A(\xi_k,\gamma)\rangle \\ \hat{R}_A^{(1)} &= \langle A(\xi,\gamma)\rangle = m_A(\xi) \end{aligned} \tag{39}$$

The correlation functions of the successive approximations are as follows:

$$\begin{aligned} \hat{R}_0(x_1,x_2) &= 0 \\ \hat{R}_1(x_1,x_2) &= \langle Y_1(x_1,\gamma)Y_1(x_2,\gamma)\rangle \\ &= \langle[\varphi(x_1,\gamma)+\int_0^l K(x_1,\xi_1,\gamma)\varphi(\xi_1,\gamma)\,\mathrm{d}\xi_1] \\ &\quad \times[\varphi(x_2,\gamma)+\int_0^l K(x_2,\xi_2,\gamma)\varphi(\xi_2,\gamma)\,\mathrm{d}\xi_2]\rangle \\ &= \langle\varphi(x_1,\gamma)\varphi(x_2,\gamma)\rangle+\int_0^l\langle K(x_1,\xi_1,\gamma)\rangle\langle\varphi(\xi_1,\gamma)\varphi(x_2,\gamma)\rangle\,\mathrm{d}\xi_1 \\ &\quad +\int_0^l\langle K(x_2,\xi_2,\gamma)\rangle\langle\varphi(x_1,\gamma)\varphi(\xi_2,\gamma)\rangle\,\mathrm{d}\xi_2 \\ &\quad +\int_0^l\int_0^l\langle K(x_1,\xi_1,\gamma)K(x_2,\xi_2,\gamma)\rangle\langle\varphi(\xi_1,\gamma)\varphi(\xi_2,\gamma)\rangle\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &= \hat{R}_\varphi(x_1,x_2)+p^2\int_0^l m_A(\xi_1)G(x_1,\xi_1)\hat{R}_\varphi(x_2,\xi_1)\,\mathrm{d}\xi_1 \\ &\quad +p^2\int_0^l m_A(\xi_1)\hat{R}_\varphi(x_1,\xi_1)G(x_2,\xi_1)\,\mathrm{d}\xi_1 \\ &\quad +p^4\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2)\hat{R}_\varphi(\xi_1,\xi_2)G(x_1,\xi_1)G(x_2,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \end{aligned} \tag{40}$$

$$
\begin{aligned}
\hat{R}_2(x_1,x_2) &= \langle Y_2(x_1,\gamma)Y_2(x_2,\gamma)\rangle \\
&= \hat{R}_\varphi(x_1,x_2)+p^2\int_0^l m_A(\xi_1)\hat{R}_\varphi(x_2,\xi_1)G(x_1,\xi_1)\,\mathrm{d}\xi_1 \\
&\quad +p^2\int_0^l m_A(\xi_1)\hat{R}_\varphi(x_1,\xi_1)G(x_2,\xi_1)\,\mathrm{d}\xi_1 \\
&\quad +p^4\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2)\hat{R}_\varphi(x_2,\xi_2)G(x_1,\xi_1)G(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
&\quad +p^4\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2)\hat{R}_\varphi(\xi_1,\xi_2)G(x_1,\xi_1)G(x_2,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
&\quad +p^4\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2)\hat{R}_\varphi(x_1,\xi_2)G(x_2,\xi_1)G(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\
&\quad +p^6\int_0^l\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2,\xi_3)\hat{R}_\varphi(\xi_2,\xi_3)G(x_1,\xi_1)G(\xi_1,\xi_2)G(x_2,\xi_3)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \\
&\quad +p^6\int_0^l\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2,\xi_3)\hat{R}_\varphi(\xi_1,\xi_3)G(x_1,\xi_1)G(\xi_2,\xi_3)G(x_2,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3 \\
&\quad +p^8\int_0^l\int_0^l\int_0^l\int_0^l \hat{R}_A(\xi_1,\xi_2,\xi_3,\xi_4)\hat{R}_\varphi(\xi_2,\xi_4)G(x_1,\xi_1)G(\xi_1,\xi_2)G(x_2,\xi_3) \\
&\quad \times G(\xi_3,\xi_4)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,\mathrm{d}\xi_4. \qquad (41)
\end{aligned}
$$

Of course, the autocovariance function is given as

$$R_n(x_1,x_2) = \hat{R}_n(x_1,x_2)-m_n(x_1)m_n(x_2)$$

and

$$\operatorname{var}[Y_n(x)] = R_n(x,x).$$

In the example considered, the quantity η occurring in the error estimate (21) is

$$
\begin{aligned}
\eta &= \sup_{x\in[0,1]}\int_0^1 \|K(x,\xi,\gamma)\|_{L_2}\,\mathrm{d}\xi \\
&= p^2\sup_x\int_0^1 \|G(x,\xi)A(\xi,\gamma)\|_{L_2}\,\mathrm{d}\xi \\
&= p^2\sup_x\int_0^1 \left\{\int_\Gamma |G(x,\xi)A(\xi,\gamma)|^2\,\mathrm{d}P(\gamma)\right\}^{1/2}\mathrm{d}\xi \\
&= p^2\sup_x\int_0^1 |G(x,\xi)|\left\{\int_\Gamma |A(\xi,\gamma)|^2\,\mathrm{d}P(\gamma)\right\}^{1/2}\mathrm{d}\xi \\
&= p^2\sup_x\int_0^1 |G(x,\xi)|\,\mathrm{d}\xi(\sigma_A^2+\mathrm{m}_A^2)^{1/2}.
\end{aligned}
$$

Since the basic existence and uniqueness theorem requires $\eta < 1$ we obtain the following condition for the applicability of the method

$$p^2 \bar{G} v < 1, \quad \text{or} \quad p^2 v < 1/\bar{G} \tag{42}$$

where p is a frequency of the harmonic excitation of the beam, and

$$\bar{G} = \sup_{x \in [0,1]} \int_0^1 |G(x,\xi)| \, \mathrm{d}\xi,$$

$$v = (\sigma_A^2 + m_A^2)^{1/2} = m_A \left(1 + \left(\frac{\sigma_A}{m_A} \right)^2 \right)^{1/2}. \tag{43}$$

For the Green's function given by (29)

$$\int_0^1 |G(x,\xi)| \, \mathrm{d}\xi = \frac{x}{24}(x^3 - 2x^2 + 1)$$

and

$$\bar{G} = \sup_x \int_0^1 |G(x,\xi)| \, \mathrm{d}\xi = \tfrac{5}{384}.$$

Of course, the factor $\|\varphi\|$ occurring in the error estimate (21) is

$$\|\varphi\| = \sup_{x \in [0,1]} \|\varphi(x,\gamma)\|_{L_2}.$$

In our case, when Q is assumed to be deterministic φ is also deterministic and

$$\|\varphi\|_{L_2} = \int_0^1 Q(\xi) G(x,\xi) \, \mathrm{d}\xi. \tag{44}$$

In the particular case when $Q(x) = Q_1 = \text{const.}$

$$\|\varphi\| = Q_1 \bar{G}.$$

Estimate (21) remains valid if

$$Q_1 = \sup_{x \in [0,1]} Q(x).$$

Particular case. To make further analysis effective, one should assume specific forms of the random function $A(x,\gamma)$ and the right-hand side term $Q(x)$. Such special forms can characterize various situations of practical interest. Here, for illustrative purposes, we assume that $\varphi(x)$ is deterministic and that $A(x,\gamma)$ is a product of a smooth deterministic function and random variable, i.e.

$$A(x,\gamma) = f(x) \cdot A_1(\gamma). \tag{45}$$

Such a form for $A(x,\gamma)$ may characterize the so-called "specimen-to-specimen" randomness in deterministically varying mass density.

Based on eqn (45), we have

$$m_A(x) = m_{A_1} f(x)$$

$$R_A(x_1, x_2) = \sigma^2_{A_1} f(x_1) f(x_2)$$

$$\hat{R}_A(x_1, x_2) = (\sigma^2_{A_1} + m^2_{A_1}) f(x_1) f(x_2).$$

Assuming, for simplicity, that $\varphi(x)$ is deterministic, the mean values of the successive approximations are:

$$\begin{aligned} m_n(x) &= \langle Y_n(x,\gamma)\rangle = \varphi(x) + b_1 c_1(x) + b_2 c_2(x) + \cdots + b_n c_n(x) \\ m_0(x) &= \langle Y_0(x,\gamma)\rangle = \varphi(x), \quad n = 1, 2, \ldots \end{aligned} \tag{46}$$

where

$$b_n = p^{2n}, \tag{47}$$

$$c_n(x) = \langle A_1^n(\gamma)\rangle \int_0^1 f(\xi_n) G(x, \xi_n) \int_0^1 f(\xi_{n-1}) G(\xi_n, \xi_{n-1}) \ldots \int_0^1 f(\xi_1) G(\xi_1, \xi) \times \int_0^1 Q(\xi_0) G(\xi_1, \xi_0)\, \mathrm{d}\xi_0\, \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{n-1}\, \mathrm{d}\xi_n. \tag{48}$$

The general formula for $R_n(x_1, x_2)$, $n = 1, 2, \ldots$ has the form

$$R_n(x_1, x_2) = \sum_{i,j=1}^{n} (b_{i+j} - b_i b_j) c_i(x_1) c_j(x_2). \tag{49}$$

where

$$R_0(x_1, x_2) = 0.$$

Numerical example. To illustrate numerically the proposed approach, we assume specific values for the problem parameters. Here, A_1 is taken to be a log–normal random variable with mean and coefficient of variation given by

$$\langle A_1(\gamma)\rangle = \mu_{A_1} = \pi^2$$

$$\delta_{A_1} = \frac{\sigma_{A_1}}{\mu_{A_1}} = \frac{1}{10}.$$

These values correspond to a flexible beam whose nominal fundamental period is 0.5 Hz. The nth moment of $A_1(\gamma)$ required in (48) can then be written as

$$\langle A_1^n(\gamma)\rangle = \{\mu_{A_1}(1 + \delta^2_{A_1})^{n-1/2}\}^n. \tag{50}$$

We take the spatial variation in the density of the beam to be of the form

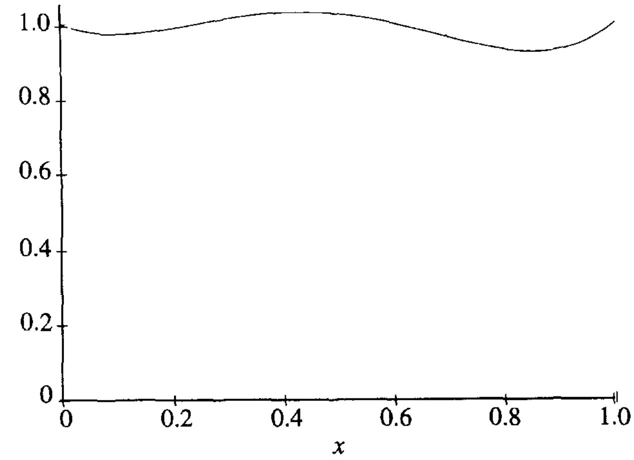

Fig. 1. Deterministic variation in mass density, $f(x)$.

$$f(x) = \frac{20x^4 - 37x^3 + 20x^2 - 3x + 5}{5}.$$

This function is shown graphically in Fig. 1. We further assume that the function $Q(x) = 1$.

Figure 2 shows the first four approximations for the mean amplitude of the harmonic response of the beam for $p = 1.25$. The convergence is seen to be excellent. Similarly, Fig. 3 shows that the standard deviation of the amplitude converges rapidly to the solution.

The mean amplitude at the midpoint of the beam is shown in Fig. 4 as a function of the frequency p. Notice that the solution convergence slows as the frequency increases. This result is in agreement with the applicability range specified by (42). Finally, Figs 5 and 6 show the mean and standard deviation of the amplitude of the response as a function of the coefficient of variation of $A_1(\gamma)$.

4.2. *Beam on a randomly varying Winkler foundation*

As a second example, we consider a finite length beam with deterministic properties resting on a randomly inhomogeneous elastic foundation. As indicated previously, $A(x, \gamma)$ is taken as the difference $A(x, \gamma) = A_b - A_f(x, \gamma)$, where A_b is a deterministic constant characterizing the properties of the beam and $A_f(x, \gamma)D$ describes the random inhomogeneity of the foundation. In this case, the governing differential equation is

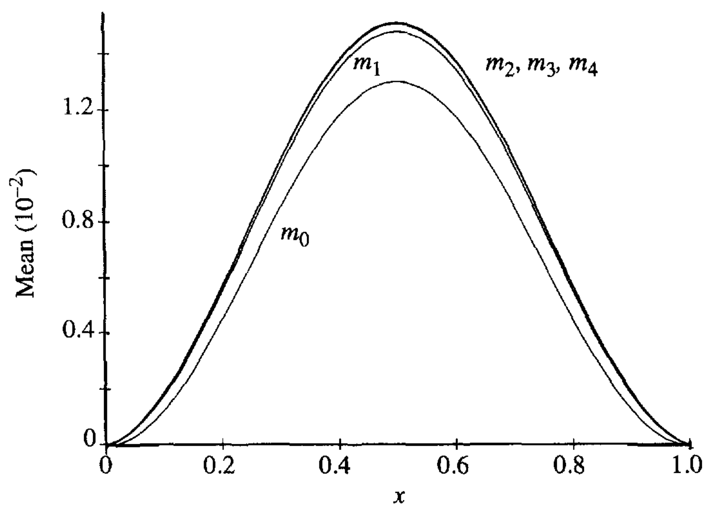

Fig. 2. Mean of amplitude, $p = 1.25$.

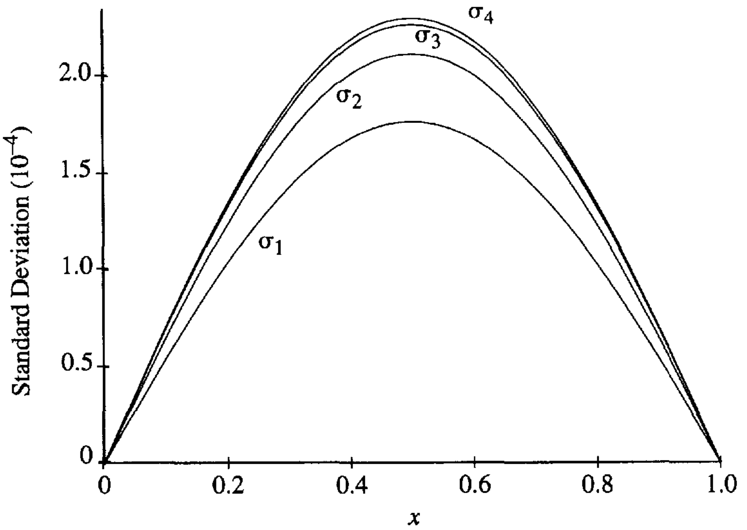

Fig. 3. Standard deviation of the response amplitude, $p = 1.25$.

$$\frac{\partial^4 w(x,t)}{\partial x^4} + A_b \frac{\partial^2 w(x,t)}{\partial t^2} + A_f(x,\gamma) w(x,t) = q(x,t) \tag{51}$$

where $q(x,t)$ includes factor D^{-1},

$$A_b = hD^{-1}\rho, \quad A_f(x,\gamma) = D^{-1}k(x,\gamma) \tag{52}$$

and $k(x,\gamma)$ is the stiffness of the random foundation. The boundary conditions for the beam are

$$\left.\frac{\partial^2 w}{\partial x^2}(x,t)\right|_{x=0,1} = 0, \quad \left.\frac{\partial^3 w}{\partial x^3}(x,t)\right|_{x=0,1} = 0. \tag{53}$$

For the case of harmonic vibration, we assume a solution of the form $w(x,t) = Y(x)\sin pt$ to obtain

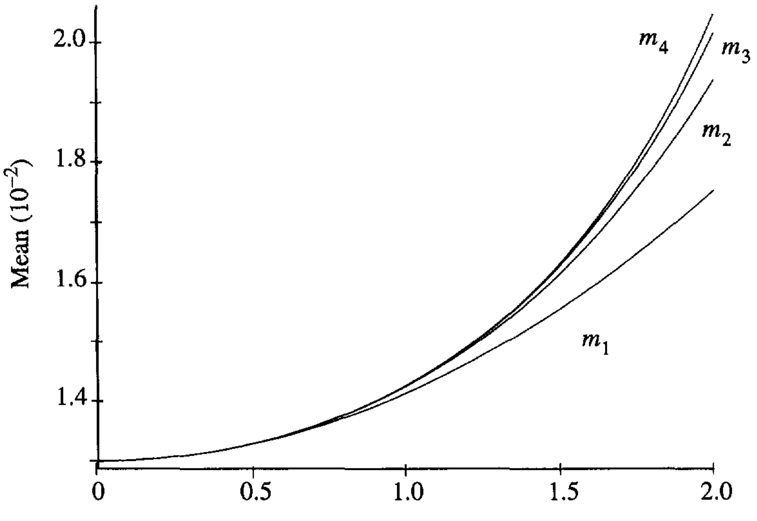

Fig. 4. Mean response vs excitation frequency p at $x = 0.5$.

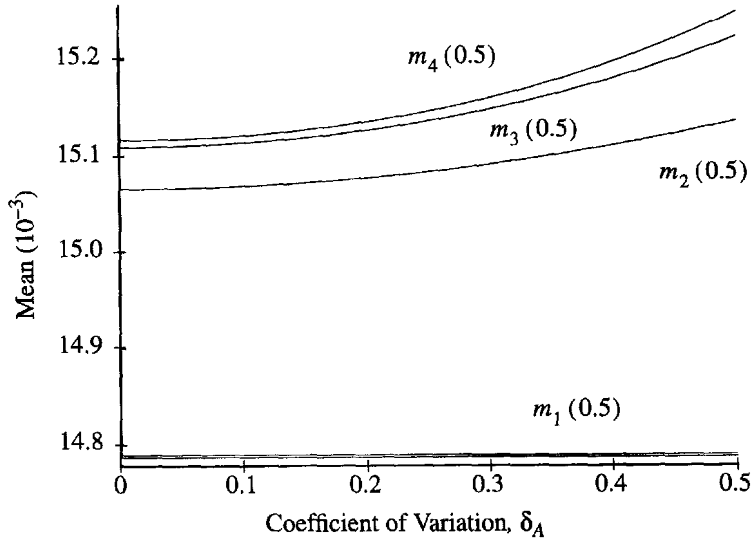

Fig. 5. Mean response vs mass density coefficient of variation at $x = 0.5$, $p = 1.25$.

$$LY - \{p^2 A_b - A_f(x, \gamma)\} Y = Q(x) \tag{54}$$

with the boundary conditions (53), where $L = \partial^4/\partial x^4$. In this example, we assume that the spatial randomness in the foundation properties can be characterized by a random field with a constant mean value, i.e.

$$A_f(x, \gamma) = m_{A_f} + A_f'(x, \gamma)$$

where $A_f'(x, \gamma)$ characterizes the random fluctuation of the processes about its mean, m_{A_f}. Introducing the new operator

$$L' = L - p^2 A_b + m_{A_f} \tag{55}$$

the integral representation of Eq. (54), along with the boundary conditions, is:

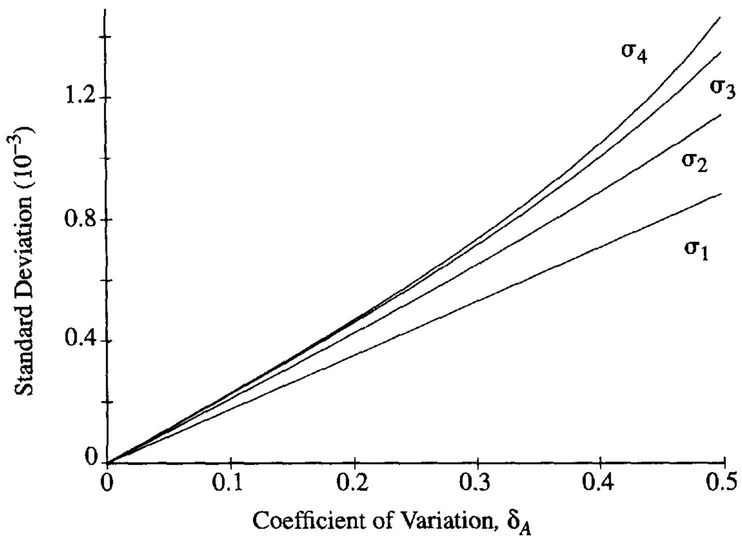

Fig. 6. Standard deviation of the response vs mass density coefficient of variation at $x = 0.5$, $p = 1.25$.

$$Y(x) = \varphi(x) + \int_0^1 K(x, \xi, \gamma)\, Y(\xi)\, \mathrm{d}\xi \tag{56}$$

where

$$\varphi(x) = \int_0^1 Q(\xi) G(x, \xi)\, \mathrm{d}\xi \tag{57}$$

$$K(x, \xi, \gamma) = -A_f'(x, \gamma) G(x, \xi) \tag{58}$$

and $G(x, \xi)$ is the Green's function associated with the operator L' and the specified boundary conditions. For this operator, the Green's function can be readily determined via a symbolic algebra computer program such as Maple (1991).

We assume that the random function $A_f'(x, \gamma)$ has a covariance function $R_{A_f'}(x_1, x_2)$. Making use of the general ideas described in Sections 3 and 4.1 one can calculate the approximations for the mean and covariances of the response.

Numerical results. To numerically illustrate the results, we assume that $A_b = \pi^2$ is a deterministic constant, and $A_f(x, \gamma)$ is a log-normal random process with mean and correlation function given by

$$\langle A_f(x, \gamma) \rangle = m_{A_f} = \pi$$

$$R_{A_f}(x_1, x_2) = \frac{\pi^2}{100} \exp\{-\alpha(x_2 - x_1)^2\}.$$

The nth order correlation function for the process $A_f(x, \gamma)$ can be shown to be

$$R_{A_f}^{(k)}(\xi_1, \xi_2 \ldots, \xi_k) = \langle A_f(x_1, \gamma) A_f(x_2, \gamma) \ldots A_f(x_p, \gamma) \rangle$$
$$= \exp\left(p m_Z + \frac{p}{2}\{\sigma_Z^2 - (p-1) m_Z^2\} + \sum_{i=2}^{p} \sum_{j=1}^{i-1} R_Z(x_i - x_j)\right) \tag{59}$$

where

$$m_Z = \ln \frac{m_{A_f}}{\sqrt{(1 + \sigma_{A_f}^2 / m_{A_f}^2)}} \tag{60}$$

$$\sigma_Z^2 = \ln(1 + \sigma_{A_f}^2 / m_{A_f}^2) \tag{61}$$

$$R_Z(x_2 - x_1) = \ln\{R_{A_f}(x_2 - x_1) + m_{A_f}^2\} - 2m_Z + m_Z^2 - \sigma_Z^2. \tag{62}$$

The first few correlation functions for $A_f'(x, \gamma)$ are then given by

$$R_{A_f'}(x_1, x_2) = R_{A_f}(x_1, x_2) - m_{A_f}^2 \tag{63}$$

$$R_{A_f'}^{(3)}(x_1, x_2, x_3) = R_{A_f}^{(3)}(x_1, x_2, x_3)$$
$$- m_{A_f}\{R_{A_f}(x_1, x_3) + R_{A_f}(x_1, x_2) + R_{A_f}(x_2, x_3)\} + 2m_{A_f}^3 \tag{64}$$

$$R_{A_f}^{(4)}(x_1,x_2,x_3,x_4) = R_{A_f}^{(4)}(x_1,x_2,x_3,x_4) - m_{A_f}\{R_{A_f}^{(3)}(x_1,x_2,x_3) + R_{A_f}^{(3)}(x_1,x_2,x_4) + R_{A_f}^{(3)}(x_1,x_2,x_3) + R_{A_f}^{(3)}(x_2,x_3,x_4)\} m_{A_f}^2\{R_{A_f}(x_1,x_2) + R_{A_f}(x_1,x_3) + R_{A_f}(x_1,x_4) + R_{A_f}(x_2,x_3) + R_{A_f}(x_2,x_4) + R_{A_f}(x_3,x_4)\} - 3m_{A_f}^4. \quad (65)$$

Let us also assume that

$$Q(x) = x(1-x).$$

Figures 7 and 8 present the first few approximations for the mean and the standard deviation of the response, respectively, for the case when $\alpha = 3$ and $p = 6.9$. As can be seen, the solutions converge quickly. The influence of the correlation parameter α on the standard deviation of the response is given in Fig. 9. As expected, the standard deviation of the response decreases as α increases.

In this example, the condition for the applicability of the method is thus

$$p^2\bar{G}\sigma_{A_f} < 1, \quad \text{or} \quad \sigma_{A_f} < 1/p^2\bar{G} \quad (66)$$

where p is a frequency of the harmonic excitation of the beam. Figure 10 shows a plot of the allowable standard deviation σ_{A_f} vs frequency resulting from (66). Because the beam considered is undamped, the response will be infinite at the natural frequencies of the beam/foundation system. As demonstrated in Fig. 10, at these frequencies (e.g. ~ 7 Hz and 19 Hz), the allowable standard deviation σ_{A_f} is zero.

5. CONCLUSIONS

In this paper, we have shown that a significant class of problems associated with the dynamics of structural systems with randomly varying parameters can be formulated and effectively analyzed via random integral equations. The analysis presented indicates that, within the applicable range, the sequence of successive approximations of the response is

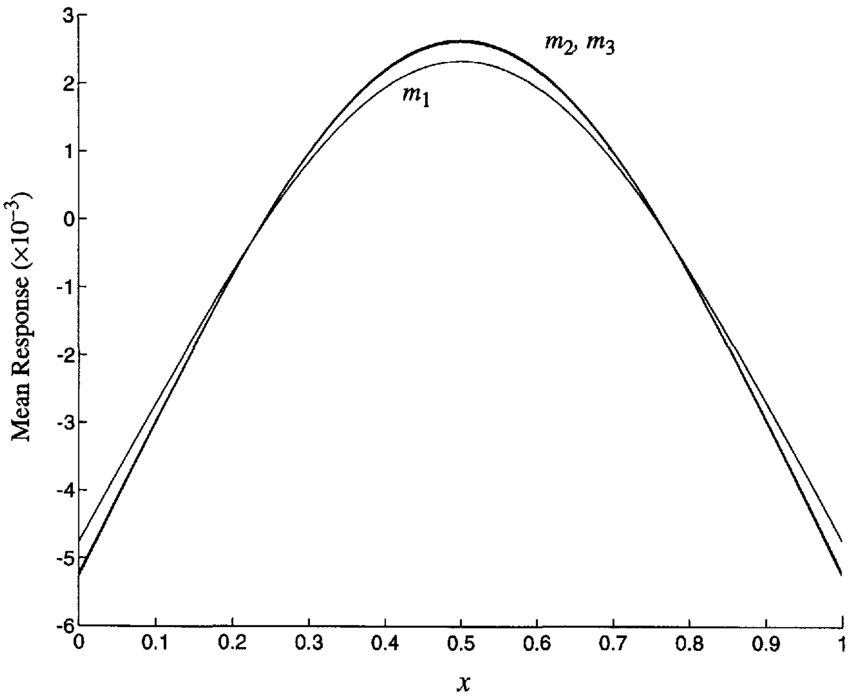

Fig. 7. Mean of response amplitude, $p = 6.9$.

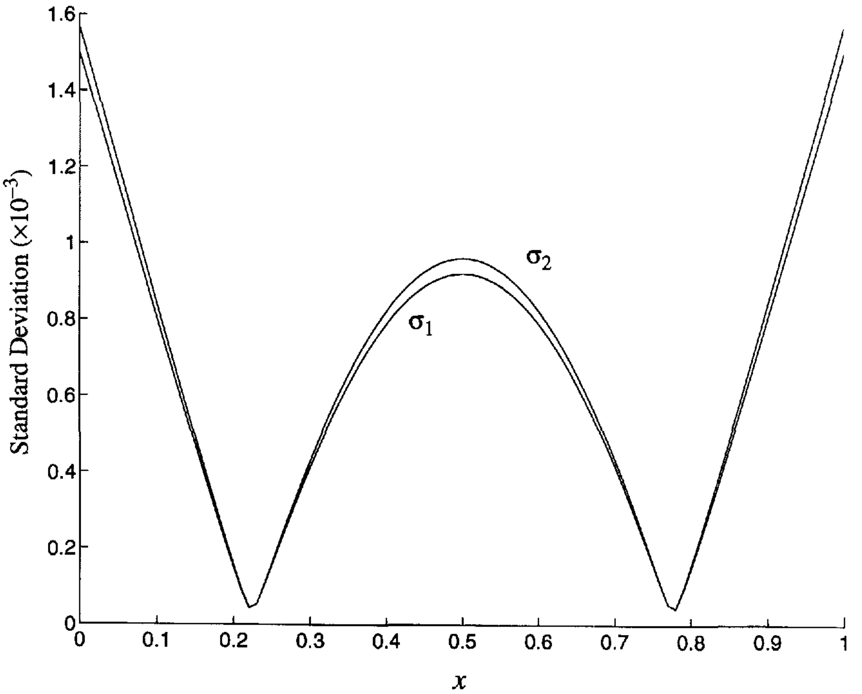

Fig. 8. Standard deviation of the response amplitude, $p = 6.9$.

convergent in the mean square sense. The formulae for the mean and covariance functions of the response can be easily obtained and used for calculations. The error estimates depend explicitly on the number of approximations and the intensity of the spatial randomness.

Application of the method to the dynamic analysis of (i) a beam with spatial randomness in the mass density and (ii) a beam resting on a randomly varying Winkler foundation shows, based on numerical calculations, that one only needs to take the first few approximations to obtain highly accurate results for the mean and variance of the response.

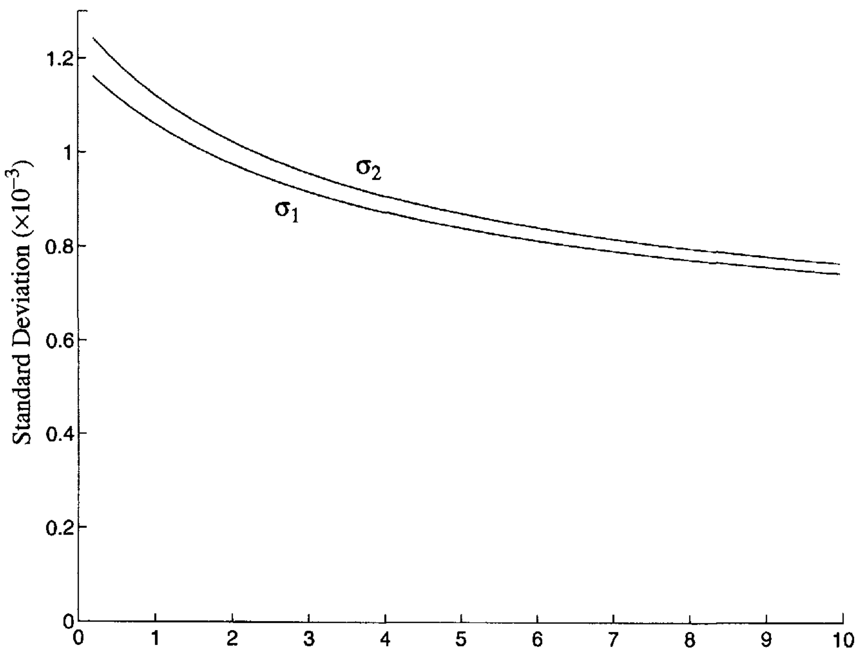

Fig. 9. Standard deviation of the response amplitude vs the correlation parameter α. $p = 6.9$.

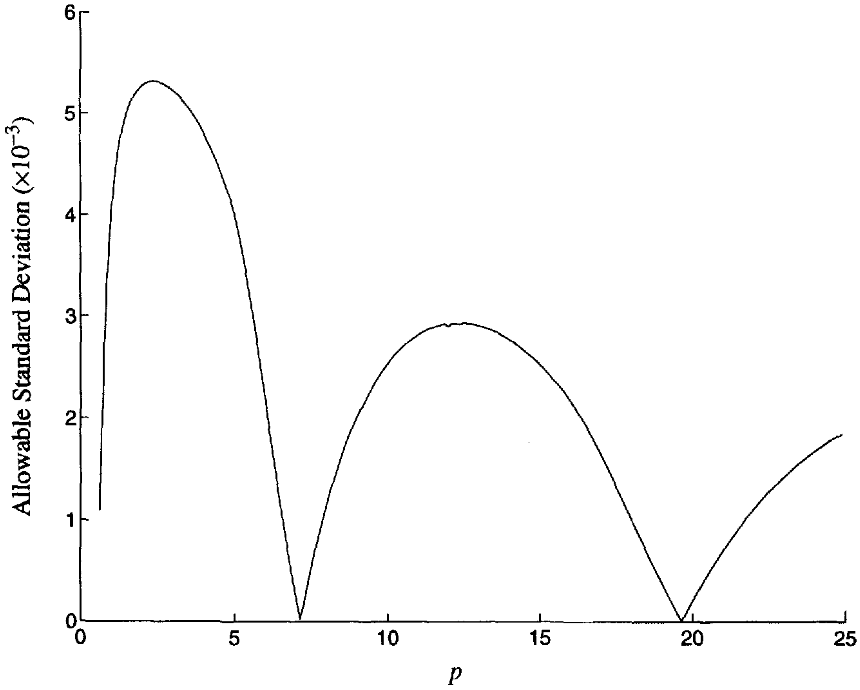

Fig. 10. Allowable standard deviation of the foundation inhomogeneity vs frequency.

The inherent features of the method (e.g. the explicit error estimate), as well as the observations stemming from solving the problems of beam dynamics, clearly indicate the advantages of random integral equations in the analysis of problems with spatial randomness; in particular, in the analysis of beams with random properties. It is worth bearing in mind these features in situations when various discretization schemes appear to be the only possible solution method.

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REFERENCES

Astil C. J., Nosseir S. B. and Shinozuka M. (1972). Impact loading on structures with random properties. *J. Struct. Mech.* **1(1).**

Bolotin V. V. (1965). On elastic deformations of underground pipes in statistically inhomogeneous soil. *Stroitielunaja Miekh., Raschiot Sooruzjenij*, No. 1 (in Russian).

Bolotin V. V. (1971). *Application of Probability Methods and Reliability Theory in Structural Analysis.* Moskwa (in Russian).

Boyce W. E. (1962). Random vibrations of strings and bars. *Proc. 4-th U.S. National Congress of Appl. Mech.*, **1,** 77.

Boyce W. E. (1967). A "dishonest" approach to certain stochastic eigenvalue problems. *SIAM J. Appl. Math.*, **15,** 143–152.

Bucher C. G. and Shinozuka, M. (1988). Structural response variability II. *J. Engng Mech.* **114(12),** 2035–2054.

Collins J. D. and Thomson W. T. (1969). The eigenvalue problem for structural systems with statistical properties. *AIAA J.* **7,** 642–649.

Deodatis G. and Shinozuka M. (1988). Stochastic FEM analysis of nonlinear dynamic problems and methods in civil engineering. *Proc. 5th ASCE Specialty Conf.* (Blacksburg), ASCE.

Fryba L., Nakagiri S. and Yoshikawa N. (1993). Stochastic finite elements for a beam on a random foundation with uncertain damping and moving force. *J. Sound Vibr.* **163(1)** 31–47.

Ghanem R. G. and Spanos P. D. (1991). *Stochastic Finite Elements: A Spectral Approach.* Springer, New York.

Iyengar R. N. and Manohar C. S. (1989). Probability distribution of the eigenvalues of random string equation, *J. Appl. Mech.* (*Trans. ASME*), **56, March.**

Kardara A., Bucher C. G. and Shinozuka M. (1989). Structural response variability III. *J. Engng Mech.* **115(8)** 1726.

Maple V Language Reference Manual (1991). University of Waterloo. Springer, New York.

Sobczyk K. (1970). Random vibrations of statistically inhomogeneous elastic systems. *Proc. Vibr. Problems,* **4(11).**

Sobczyk K. (1972). Free vibrations of elastic plate with random properties—the eigenvalue problem. *J. Sound Vibr.* **22(1),** 33–39.

Sobczyk K. (1985). *Stochastic Wave Propagation.* Elsevier-PWN.

Sobczyk K. (1991). *Stochastic Differential Equations with Applications to Physics and Engineering.* Kluwer Academic, Dordrecht.

Szynal D. and Wedrychowicz S. (1985). On existence and asymptotic behaviour of solutions of a nonlinear stochastic integral equation. *Annali di Matematica Pura et Applicta,* **142** 105–119.

Szynal D. and Wedrychowicz S. (1988). On the solution of a stochastic integral equation of the Volterra type with applications for chemotherapy. *J. Appl. Probab.* **25,** 257–267.

Tsokos C. P. and Padgett W. J. (1974). *Random Integral Equations with Applications to Life Science and Engineering.* Academic Press, New York.

Vanmarke E. H. and Grigoriu M. (1983). Stochastic finite element analysis of simple beams. *J. Engng Mech., ASCE.* **109(5).**

vom Scheidt J. and Purkert W. (1983). *Random Eigenvalue Problems.* North Holland, New York.